Challenging math problems for mathematically gifted children

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Abstract

Mathematical talent can be regarded as the result of a process based on influencing factors like genetic, environmental and individual orientated variables. We consider well designed problems as one of the environmental variables which have a great impact on the development of mathematical skills. Referring on Kießwetter's patterns of action (Kießwetter 1985), we analyze cognitive components of problem solving the children can acquire by doing mathematics. Furthermore the development of mathematical talent is based on emotional aspects as well as capabilities needed in interaction and communication processes.

Key words: Mathematical talent; patterns of action; fields of problems; problem solving

Introduction

Since school year 1999/2000, at the University of Hamburg we are fostering mathematically talented children of the third and fourth grade within the framework of our project called PriMa[1]. This project is a research project, which simultaneously aims at fostering mathematically talented children. Every year, we start with a talent search attended by around 400 children. Out of the 400 children, we usually invite approximately 50 for working on mathematical problems at the university for a period of time until the end of the fourth grade. By that, during the last twelve years, since the project has been established, we have fostered about 600 children (eight to ten years old).

The following considerations are based on our long-lasting experiences and discussions. What are the essentials of fostering mathematical gifted children? Obviously there are several aspects to take into considerations for establishing challenging and supporting learning environments. In this article I focus on problems[2]. For us, one essential aspect is the construction of fields of problems: problems which are embedded in a rich mathematical context and offer the opportunity of asking further questions. Working on interesting and challenging problems has an essential impact on the development of cognitive, emotional and social skills. Based on examples the broad impact of well-chosen problems on developmental processes concerning mathematical talent is described. In this context cognitive components of problem solving, patterns of action (Kießwetter 1985) are explained.

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[1] PriMa is a cooperation project of the *Hamburger Behörde für Schule und Berufsbildung* (u.a. *BbB*), and the *William-Stern* Society (Hamburg), the University of Hamburg. (for further information you are invited to visit the website http://blogs.epb.uni-hamburg.de/nolte/)

[2] In Germany in math education "problem" means mathematical problem, not necessarily reality based questions.

About the development of high mathematical abilities

Referring to approved models of giftedness e.g. Munich Model of Giftedness (Heller 2004) and Gagné's Differentiated Model of Giftedness and Talent (Gagné 2004) mathematical talent can be regarded as the result of a process based on influencing factors like genetic, environmental and individual orientated variables. Additionally Ziegler and Phillipson (2012) refer to a systemic approach asking for the interdependence of the involved factors. These multidimensional approaches can be applied also to mathematics education and the development of high mathematical abilities.

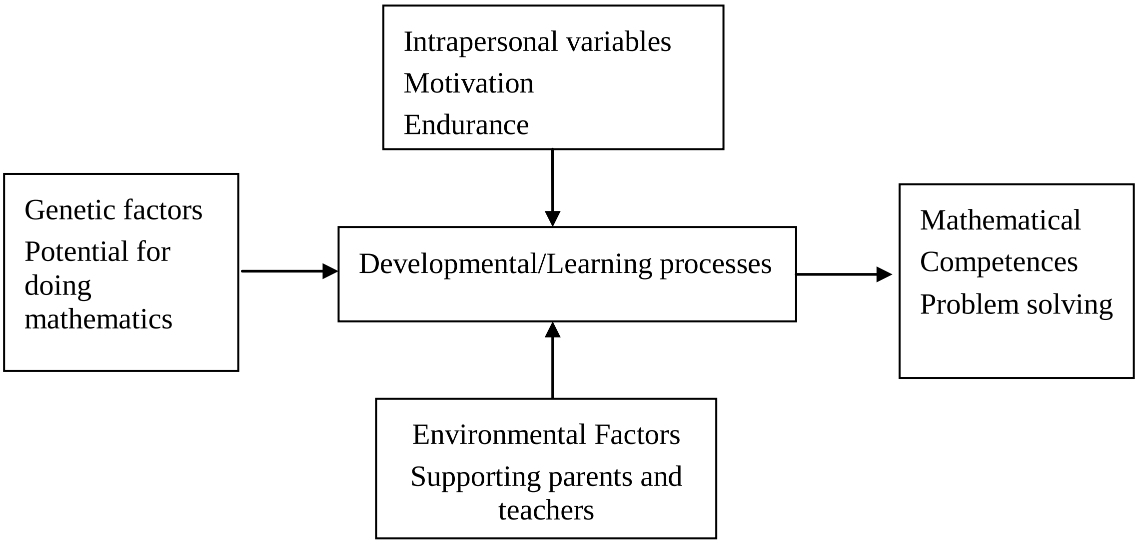


Fig. 1: Development of mathematical competences

Based on genetic factors developmental and learning processes lead to competence/skills. The way to gather those, the amount of skills and also the character of the skills are influenced by activities of the learners. On the learners site important aspects are for example motivation and endurance (Nolte 2006; Lucas and Claxton 2010) refer to self-discipline as a determining criterion for successful performance. If learners have the same IQ self-discipline makes a difference concerning performance (Lucas and Claxton 2010, p 13). Also research about genetics and developmental processes particularly in the field of high abilities show the interplay between genetic factors and motivation: „The substantial genetic influence at the high end of the distribution suggests that engaging in deliberate practice is in part a function of genes influencing ability indirectly, but powerfully, through motivation. Put more simply, genes code for appetites, not just aptitudes." (Kovas, Haworth et al. 2007, p 107). It lays in the responsibility of parents and well-educated teachers to appease the appetite for mathematics or at least support their children by showing interest in their activities. In this model parents, teachers and peers are part of environmental aspects.

Used in fostering projects for mathematically gifted children the described model can be further modified. In this context the starting point lays in previous knowledge of the children and in a general potential for mathematics learning. Important environmental factors are well composed groups, tutors who are qualified in mathematics as well as in teaching mathematics. We regard choosing challenging and well considered selected problems as one important factor as well.

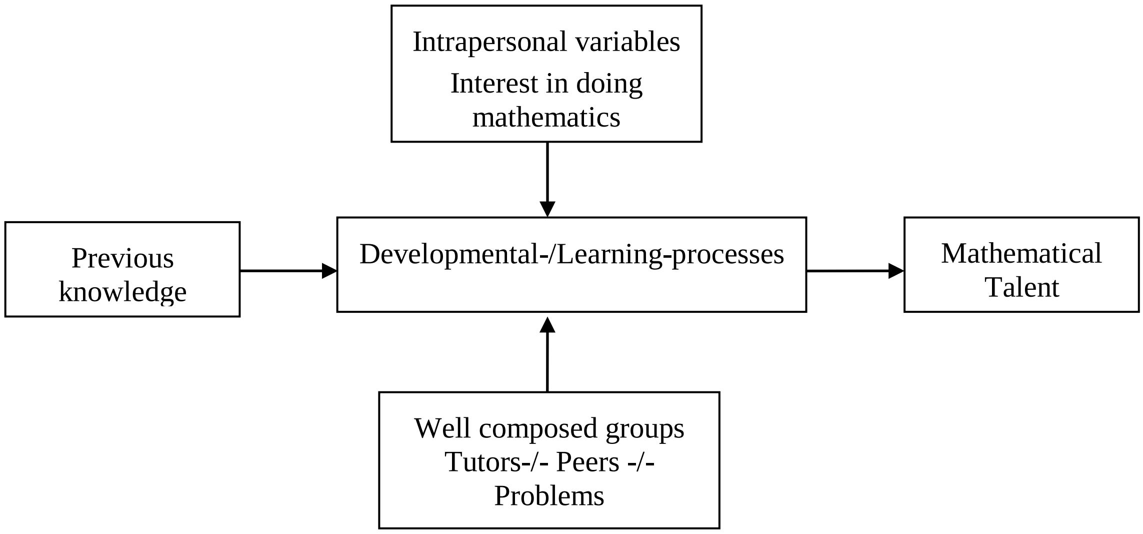


Fig. 2: Some aspects of development of mathematical talent

But what is mathematical talent? It differs, depending on the age of the children and also on the level of mathematical knowledge achieved. We all have an idea, which is based on experiences in normal development and on curricula. Interests of gifted children can be regarded as leading factor for improvement of mathematical knowledge. Therefore the development of mathematical skills depends deeply on the activities of the children. When they start school not only the level of acquired knowledge differs, but also in between one child knowledge can be at a different level. Johannes was almost six, when he could tell the preceding and succeeding number of any number up to 1,000,000. He was also able to read accurately large numbers (in digit writing). He loved to add three digit numbers but could hardly multiply (Nolte 2012). Due to the fact that mathematical talent is changing during the process of development and also depending on situation and environment, an approach to describe mathematical talent lies in a shift from identifying them to a description of the competences which must be acquired.

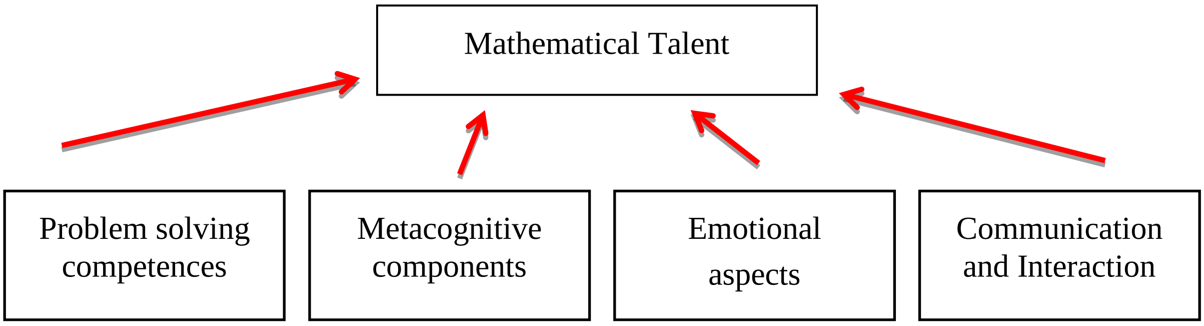


Fig. 3: Essential components of mathematical talent

This report will focus on the effects of problems on learning, on motivation, on acquisition of heuristics.

Mathematical problems and the development of mathematical, cognitive, emotional and social skills

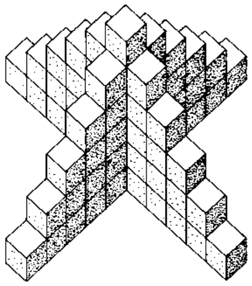
In our point of view, one of the most important aspects of our project is supporting children by giving them appropriate tasks. We encourage them to do their own investigations. Doing research while learning how to do research is one of our main goals. Therefore problems are designed with rich mathematical contents, which allow children to work on them as if they were little researchers. Most of the problems are open problems and offer the possibility of posing further leading questions. These kinds of problems are called field of problems. The problems do not require more than usual knowledge, which children at this age are expected to have. Nevertheless they are challenging due to their complexity.

Before talking more about theoretical aspects here is one example:

Example

The “Tower Task“ is a task we assign to 8-year old children:

Skeleton Tower [3]



Questions

(i) How many cubes are needed to build this tower?

(ii) How many cubes are needed to build a tower like this, but 12 cubes high?

(iii) Explain how you worked out your answer to part (ii).

(iv) How would you calculate, the number of cubes needed for a tower *n* cubes high?

Fig. 4 The tower task

This task is normally designed for older children: “It is provided for the upper half of the ability range and designed for children age 13-16, but is more widely applicable with some adaptation.”[4] The children in our groups do not need any adaptation.

They found different ways of solving the problem as these examples show:

- Counting the cubes in the different walls as well as those of the middle „pillar“: 4 x (1+2+3+4+5) + 6
- Counting levels: 1 + 5 + 9 + 13, …
- Two walls are put on top of each other. You get two cuboids: 6x6 and 5x6
- The two side wings form a cuboid when folded down on the remaining wall: 11 x 6

Particularly gifted children are able to look at such a shape from different perspectives and then e.g. make a single wall out of four. They do this through visualization. They see patterns and structures, which they use to simplify the picture at hand. They can generalize their ideas. It is also important to note that they work on a symbolic level, i.e. they do not use real cubes to replicate the towers. Some children come up so quickly with a solution to the first question that they actually seem to „see“ it. They instantly record the sum of entities. In doing so, they draw back on calculation processes, i.e. they have factual knowledge at hand in a very fast way.

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[3] Shell Centre for Mathematical Education (1984). Problems with Patterns and Numbers. Masters for Photocopying.(p 5 (18))

[4] http://www.mathshell.com/scp/ppn51.htm

To work problems like these the children can use different strategies. One of the first could be counting. Even counting is not meant to be easy because you have to complement the unseen cubes. Besides the fact that this should be done systematically, counting alone does not lead to any generalizing result. Seeing patterns and structures facilitates the solving process. Furthermore this is a basis for generalization.

This problem can be called a complex problem. The children have to recognize patterns and structures as well on the geometric level as on an arithmetic level. The building itself is complex and allows several perspectives on its structure. Beyond that also the calculation of towers of different sizes is difficult for the children. Normally they are not experienced in the necessary calculation processes. Also during this part of the problem solving process, recognizing patterns and structures allows or simplifies the way of getting a solution.

Patterns of action and cognitive components

Recognizing patterns and structures in mathematical contents more often is observed with gifted children. Kießwetter[5] (1985; 2006) refers in his research to this as one of special patterns of action, which are proved to be essential in the field of mathematical problem solving. Some of these patterns of action are similar to those Krutetskii (1976) described as characteristics for mathematical giftedness, such as the ability to generalize or of reversal thinking. There are further patterns of action described by Kießwetter:

- Switching between levels of representation
- Finding connected problems
- Organizing material in order to recognize (eventually different) patterns
- Developing and testing of hypotheses
- Reduction of complexity through meta-symbolization (building super signs (chunking))
- recursion
- Intuitive use of strategies already indicated or further heuristic strategies

He said that these were advantageous in problem solving processes (Kießwetter 1985; 2006; Nolte 1999).

Depending on the characteristics of the problem, different cognitive components are needed (Nolte 2006). Comparable considerations are found in discussions about the tasks given in intelligent tests. Referring to Sternberg (1986) and Waldmann and Weinert (1990) even the activation of components like building analogies, finding patterns, inductive thinking, which are meant to prove general factors of intelligence depends on the character of the problem (Sternberg 1986; Waldmann and Weinert 1990).

To work the tower task the children especially need patterns of action such as

- recognizing patterns and structures: *sum of natural numbers*
- building super signs (chunking): *seeing two walls together as one cuboid*
- switching between modes of representation: *geometric versus arithmetic*
- switching between arithmetic representations: *1 + 5 + 9 + 13… = 1x1, 4x1+1, 4x2+1, 4x3+1, …*
- capability to generalize: *I must build the sum of the natural numbers up to the size of the tower minus one. Then …*

Patterns of action or cognitive components of problem solving are essential in the development of heuristic strategies and in general in the development of problem solving competences. They cannot

[5] Prof. Dr. Karl Kießwetter and his group have been running a project for fostering mathematical gifted pupils of the secondary level (between 12 and 19 years old) for nearly 30 years now. http://www.hbf-mathematik.de/

be taught like algorithms. And because the used patterns of action differ depending on the problem, working on different problems lays the foundation for creative and complex problem solving.

Communication and justification

Mathematically gifted children of 8- or 9-year-old are beginners. They work as little researchers while acquiring the techniques of problem solving. Even if they use some of the described patterns of action they normally do it without being conscious about it. In our opinion the communication between other children and between a child and a tutor are essential in the process of developing mathematical competences. Working on their own they have to explain and defend their ideas. By this the tutors also support considerations about heuristic strategies.

The Plenary discussions about the (possibly) different results and different ways of solving the problem support a growing consciousness about an individual solving process but also about heuristics in general. Children learn to present their results and learn to listen to the ideas of their peers. This is not an easy process because due to the age of the children the capabilities of verbalization are limited. Additionally they are not accustomed to express themselves about their ideas in such complex learning environments. We know that capabilities to solve a problem do not necessarily come along with fitting verbalizing capabilities. Because the children often work on different ways and about different ideas they must have in mind their own ways of thinking. In addition they must be capable of comparing them during plenary discussions with different strategies and considerations, not always formulated in an understandable way.

They also get accomplished to the necessity to argue for their ideas. By this they increase their competencies in mathematical argumentation. They offer different perspectives on the problems; they show a variety of ways of solving the problems. We support the development of "*organization material in order to recognize (eventually different) patterns*" e.g. by using tables, if this makes sense. Within this task we support the "*switching between modes of representation (geometric versus arithmetic*)" via asking them to show at the picture what they have calculated. If they write down number sequences like 1, 5, 9, 13 they must explain why they chose these numbers. Often their explanations are based on a different representations like 4x0+1, 4x1+1, 4x2+1, 4x3+1, ….. By this way within discussions about the problems children learn more and more about patterns of action and can use them with more and more ease.

This is closely connected with the capabilities of the tutors to structure the discussions and to interpret the information given by children.

Examples for discussions in individual situations

Already during the time children are working on their problems they talk with the tutors. They are invited to explain their thinking and to prove their ideas.

<u>Consciousness about generalization:</u>

Tutor: If you know the results for the tower of six cubes high and that of twelve cubes high, what comes next?

Lena: How to find out very quickly, if there are 80 for example.

This can be regarded as a generalization via 80 cubes as an example for any number.

<u>Questioning of hypothesis</u>

Murat is working on the question how to find out the result for the tower of height of 12

Tutor: How did you find out your result?

Murat: I doubled the result (of height of 6 M.N.)

Tutor: Why can you double it?

Murat: Because the height of the tower is twice.

Tutor: How can we test whether we can double it?

Murat: We calculate for two.

Comparing the results for height one and height two the child realizes, that doubling does not work.

Leonard wrote: 2+98+3+97+…+49+51=48x100

Tutor: Which numbers are you calculating?

Leonard: From 1 to 99. Oh, the 50 is missing.

Tutor: Now you can consider whether this works with any heights /sizes?

Especially the experience to explain and to defend their ideas is new to the children. Although in the scientific discussion of math education problem solving, arguing and proofing are important competences children have to acquire. Nevertheless especially young children in primary schools are not used to them. The interactions with the tutors as well as the plenary discussion lead the children to get used to the idea of scrutinizing their thoughts.

Habits of mind

The following example shows that children acquire explaining their thoughts and proving them as a habit of mind (see Goldenberg (1996) in Leikin (2007 p 2333). Explaining and proving become natural parts of problem solving processes.

Peter and Paul, both 9 years old and in their second year of fostering, were talking together about their results:

Peter: “I can explain to you exactly what I have done!”

Paul: “Oh, but I can also prove it!”

These thoughts laid a focus on the development of the children. Another important aspect lays in the difficulties of understanding them. Due to the age of the children, the low experience to talk about mathematics at such a high level and the still low level of language the problem of misunderstandings between children and tutors arise. Consciousness of this fact is necessary for the tutors. In most of the teaching processes you can observe a tendency to interpret a given answer of a child on the basis of the expectations of the teacher. In a sense of helping in a wrong way misunderstandings are continued.

This will be explained with the next example.

Example

The following problem is the first part of a field of problems (see Pamperien 2004). We present it to the very beginners during our talent search process.[6] Starting parts of the natural numbers are placed in a special (triangle) structure: every row contains one number more than the preceding one.

Thus, the children can recognize several patterns:

- On the right side, the sequence of the triangle numbers
- On the left side, a sequence +1, +2, +3 ...

[6] We got the idea for this problem from a task presented by Linda Jensen Sheffield (1999) at the first conference of the international group for mathematical creativity and giftedness in Münster. She used a triangle with odd numbers until 19.

- In the middle a sequence +4, +8, +12 ...

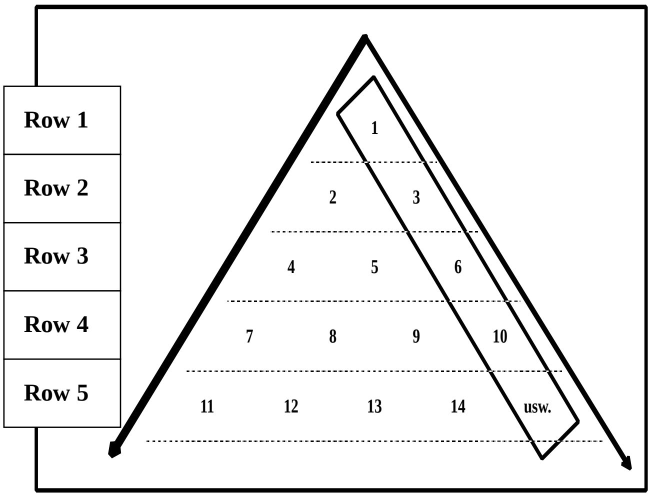

Fig. 5 The sequence of the triangle numbers

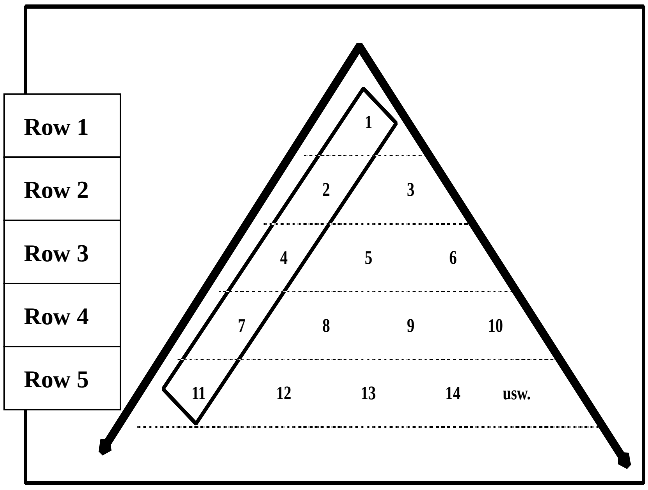

Fig. 6 The sequence of +1, +2, +3 …

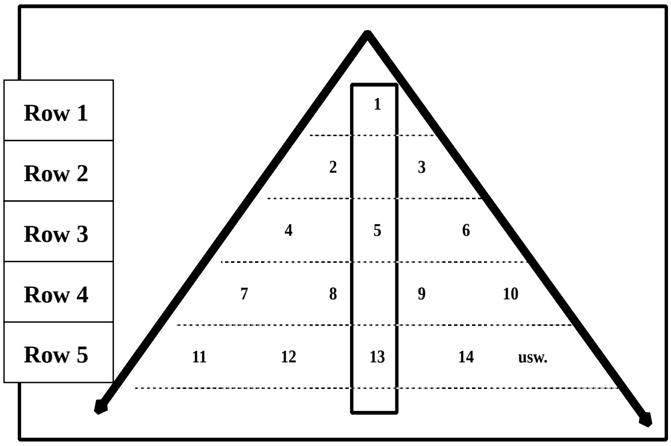

Fig. 7 The sequence of +4, +8, +12, …

Beside this, we want to know whether the children are able to orientate themselves in a system of co-ordinates. For this, we invented the names row and place.

<u>Question 1</u>: Where is the place of the number 23?

To reduce the effort of writing, the children only have to complete the sentence: *“Number 23 is found in row … at place ….”*

<u>Question 2</u>: Which number is placed in row 8 at place 8?

Question 2 refers to the reversal of the question: *“Which number is then found at place 8?”* to which the answer *“In row 8 at place row there is found number …” is pre-formulated like before.*

The final task of this part is *“Write down how you found this out.”*

These questions allow the children to continue the starting part by counting on or by using patterns.[7] Lina wrote: “… in the 5th row the last number is 15 and then + 5 +1= 21; this is the last

[7] As a remark, the field of this problem turned out to differentiate very good between mathematically gifted and mathematically average children. Children who succeed in finding solutions because they applied structures and patterns, usually we took in later into our project.

number in row 6; but still we don't have the 23; but soon. Therefore, it must be in the 7th row and we know about 21 that there are only 2". Jan said „I took the triangle in my mind and calculated like this 1+2+3+4+5+6+7=28 und 28-5=23".

Many of the children explained that they found the correct solution through processes of counting. First we interpreted this as counting on by writing down the numbers. Also this was not easy because not every child grasped the idea of a triangle. It is very difficult not to interpret the words of the children in a false manner. Many children said that they counted on or wrote this down. But we learned that *counting on* can mean

- Counting on and filling the missing numbers in
- Counting orientated in a certain pattern, for example counting the edges
- Counting only in mind
- Counting by writing down

And some of the children who went on with the scheme did this to prove the special pattern they saw. Therefore writing down does not mean that the child did not see the structures. For this reason we encourage them to tell us what they counted, what they have seen or to explain whether there are other ways to find the solution or to look for other ways. By this, much more children gave insight into some patterns. This was a very important insight for us (see Pamperien 2004).

Interaction und communication between tutors and children is based on → The approach of minimal help → The demand of explication and justifying → Different approaches and solutions support the children in getting used to a diversity of perspectives on a problem

Fig. 8 About interaction and communication processes

Emotional aspects

These considerations are combined with the next part, with the emotional aspects in problem solving processes.

Giving safety by offering possibilities of transferring obtained knowledge

Especially shy children and those who did not feel safe in experimenting with ideas needed the second part of our problem, the transfer to the triangle of odd numbers. The children were asked to compare the first triangle (natural numbers) with the second. Nevertheless it is possible to solve the problem as an isolated one. But if they compare the knowledge they obtained working through the first problem, they can use analogies.

The same questions as in task 1 and 2 were posed with task 3 and 4:

Question 3: „*Which number belongs to the left number 23?*"

Answer: " *To the left number 23 belongs the number ….*

Question 4: „Which number in the scheme (2) – thus on the right – is found in row 8 at place 8?

Answer: "On the right in row 8 is found number ….

The final task is "Write down how you did find out."

Triangle of the natural numbers (left) Triangle of the odd numbers (right)

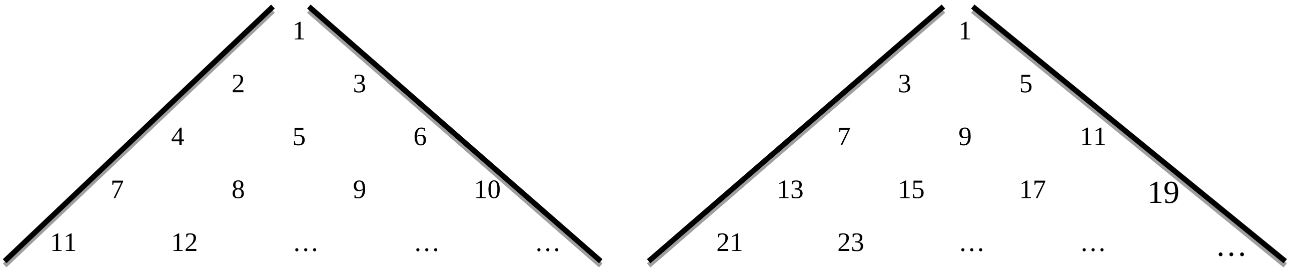

Fig. 9 Part of the working sheet

With these further leading questions the children then get the possibility to use the obtained information of the first part. Here especially patterns of action like looking for patterns can be used and applied on the second triangle. It is not only the application of newly learned strategies but furthermore the knowledge that it is allowed and wished to look for different and perhaps unusual ways. Sometimes we realize that a child is more capable than we thought while they can transfer the knowledge they obtained from one situation to another. This often happens with children who are gifted, but uncertain.

Supporting concentration and motivation by complex problems

The richness and the complexity of mathematical contents are also important for concentrating on working in mathematical contexts. Gifted children get less motivated, if the given problems are too easy to be solved. Neurobiological research shows that only appropriate challenging questions produce the necessary amount of dopamine which is fundamentally needed to ensure that the working memory functions well. Dopamine stabilizes information in working memory (Durstewitz, Kelc et al. 1999; Korte 2009). Unconscious processes evaluate given tasks about their level of difficulty. Only tasks evaluated as being at an appropriate level of difficulty lead to the release of dopamine. This means the children cannot concentrate on problems which are too easy (Korte 2009).

Supporting the development of tolerance of frustration, of endurance and volition

Another problem arises from the fact, that in regular classroom high performing children are often used to answer questions immediately. But for dealing with our kind of questions endurance and volition are needed. Due to the complexity of our fields of problems the children work on they need much more time than they usually spend on tasks at school. We work on one field of problem for about 90 minutes, some tasks require two 90-minute sessions. Therefore the designs of the problems need to support interest and motivation. So the problems facilitate successes at the beginning as well as during the process of problem solving. This is orientated on the position of Kießwetter (1985). He represents ‘motivation’ in two different aspects, the starting motivation and the process motivation. Both are essential during problem solving.

The two different motivations build a balance between success, searching processes, using wrong ways or inappropriate approaches, which are unavoidable in problem solving processes. This balance gives the necessary support for the development of endurance and volition.

Although the groups are composed from children tested as mathematically gifted, the groups are still heterogenic concerning the performance the children are capable of. Different from their experience at school, they are confronted with the experience that they are not always among the best performing pupils. Therefore the way of designing problems allows successes at different levels and different points in the solving processes also supports the concept of self-efficacy.

We esteem the ideas of the children, whether they seem to be correct or not. So we are prepared for the experience, that even long used problems sometimes are solved in a way which lay behind the experts solution space (Leikin 2007). Sometimes an idea which seems not to be conducive with some considerations of experts turns out to be a very inspiring idea.

Aspects of constructing fields of problems

In our fostering program we offer children opportunities to gather experiences with complex fields of problems. Mathematically gifted children benefit from enrichment programs. Offering contents which come later in school is only a temporarily challenge. For this reason we are looking for questions, which do not require more than usual knowledge children at this age are expected to have. The challenge comes with the complexity of the information they have to handle. The problems are open in several ways, they have: sometimes several results, several ways of finding solution and further leading questions can be found.

Opening problems is a well-known method for fostering mathematically gifted children, which for example are also found in Hashimoto and Becker (1999) and Sheffield (1999). One important question concerns the kind of opening the problems. As the children we work with are still very young and have no experience with work techniques in a complex field of problem. And there are borders regarding their working memory. Due to the limitation of the working memory we support the children by restricting the questions from the very beginning. This restriction is intended to lead the children more directly to the core of the problem, so that they do not get lost in the jungle of complexity.

Thus, it is necessary to help the children not to lose control during their solution process. One aspect is that we control their first steps. After a non-redundant introduction the process is opened for individual solution processes.

Rich comprehensive information

↓

Restricting by special examples and questions

↓

Openness during the process of problem solving
as well as
Openness for subsequent Questions

Fig. 10 Design of the problems

Also the tower task is built that way. Working on the tower task, the first question is easy to solve. However it is necessary to consider the given information. Within the second question the children can use their considerations and strategies for a broader application. Although these are guiding questions, the way of solving is open to different approaches. Only some of these approaches allow getting answers to the fourth question. We modified the last question by replacing n by 99. By using larger numbers the children are forced to use patterns and structures. They can prove their hypotheses with smaller numbers, but we avoid introducing variables. To support reversal thinking the questions end in: *We know that the tower contains 66 (153 or 91) cubes. Do you know its height?* Also we invite the children to explain their ways of solution after every question. It is easier to explain what they have done if they can use concrete examples.

(i) How many cubes are needed to build this tower?

(ii) Explain how you find out…

(iii) How many cubes are needed to build a tower like this, but 12 cubes high?

(iii) Explain how you find out…

(iv) How would you calculate, the number of cubes needed for a tower *99* cubes high?

(v) Explain how you find out…

(vi) How can you be sure …

(vii) We know that the tower contains 66 (153 or 91) cubes. Do you know its height?

Fig. 11 Modified questions of the tower task

So at the level of describing a general idea they have experienced in explaining thinking in the special mathematical context.

How do we find our questions?

There are many problems posed to foster mathematically interested and gifted children. Many of them are designed as puzzles. Seldom can we use a problem like the tower task with only small modifications. We analyze the problems taking into account the mathematical content as well as the cognitive components, which could be needed to solve the problems. By doing this we consider the variety of solutions (the solution spaces (Leikin 2007)) and variety of ways we expect the children will find. After this consideration we give the problems to the children. The next step is the evaluation of the different aspects of the problem.

Example

This is part of a problem we posed during our talent search process:

Because your neighbor has a broken leg, he cannot go dog walking. He offers you 1 cent per day you walk the dog, doubling the amount every day. Your brother says: “How stupid to accept this. I would ask for 1 € a day”. What would you do?

We expected the children to work on this task e.g. using a table:

Day	1	2	3	4	5	6	7	8	9	10	Sum
Girl	0,01	0,02	0,04	0,08	0,16	0,32	0,64	1,28	2,56	5,12	10,2
brother	1	1	1	1	1	1	1	1	1	1	10

Fig. 12 Table to organize the solving process

It is not the place to describe the whole process of developing this task. But obviously the question given to the children had to be changed:

What would you do?

Some of the children answered: “We would do it for free!”

We observe that mathematically gifted children are interested in going further with their questions. Calculation and puzzling is boring compared with the question to find rules which offers the possibility for general results. But as young as they are guidance during this process seems to make sense. Even in the way we pose the problems to the children, as if they were little researchers, are confronted with the situation, that they have to find out how to start and – what sometimes is hard to endure – when to end.

But giving them for example the Pascal triangle opens questions in so many ways that several problems arise: Questions must be posed, hypotheses developed and proved and communication becomes increasingly difficult if the underlying questions are too different.

On the contrary to let the children to solve the whole problems on their own, our guidance is necessary because we can guide them to develop higher mathematical competences.

Previous knowledge: usual knowledge at this grade	⟷	Sufficient complex
	Necessary for release of dopamine Supports endurance and Volition	
Restricting the opening questions	⟷	Allows quickly independent work
	Fast first successes support self-efficacy	
Starting motivation	⟷	Process motivation
	Successes during the process of problem solving support volition	
Ideas, results	⟷	Scrutinizing, justifying
	Support to argue	
Impulse problems	⟷	Field of mathematical problems
	Support creativity	

Fig. 13 Main aspects of the conceptual design of the fostering program

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